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The analogy between streamline curvature and buoyancy in turbulent shear flow

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A formal algebraic analogy is drawn between meteorological parameters, such as the Richardson number, and the parameters describing the effect of rotation or streamline curvature on a turbulent flow. The analogy between the *phenomena* is a good first approximation. Semi-quantitative use of the analogy to apply meteorological data to curved shear layers shows that the effects of curvature on the apparent mixing length are appreciable if the shear-layer thickness exceeds roughly 1/300 of the radius of curvature; larger effects may occur in compressible flow. Application of the Monin–Oboukhov formula considerably improves the agreement between prediction and experiment in boundary layers on curved surfaces.

1. Introduction

It is well known that certain laminar flows over curved or rotating boundaries have analogues in flows with buoyancy—for instance, in special cases the flow between two concentric rotating cylinders, and its stability to infinitesimal disturbances, can be described by the same equations as the free convection flow between a pair of parallel horizontal plates at different temperatures. Scorer (1967) has used an analogy between buoyancy and curvature in a discussion of the stability of inviscid rotating flows. The analogy between the two types of body force is not so close in turbulent flows, because buoyancy effects depend on the temperature fluctuation θ whereas curvature ('centrifugal' or Coriolis) effects depend on the circumferential velocity fluctuation u: however, the correlation coefficient between θ and u is numerically at least 0·7 in a shear flow over a slightly heated surface (Johnson 1959) so that a buoyancy/curvature analogy, like the Reynolds analogy between heat and momentum transfer, should be a good first approximation to the truth, and of practical use in the case of small body forces. Some parallels can be drawn between the dimensionless parameters in the two cases, no matter how large the body forces; apparently this was first done by Prandtl (1930, reprinted 1961) using classical mixing-length arguments, since rediscovered separately by meteorologists and workers on curved flows. (I am grateful to Prof. G. L. Mellor for drawing my attention to Prandtl's paper.) Little use seems to have been made of the analogy for turbulent flow, the only reference known to me being the paper by Thomas & Townsend (1957), who use the known properties of turbulent flow between rotating cylinders to illuminate an account of thermal convection.

In § 2 of the present paper, the analysis customarily used to derive buoyancy parameters, like the Richardson number, from the equations of motion is used to derive equivalent parameters for curved flow; the correspondence between the two sets of parameters (and a third set for flows rotating about a spanwise axis) seems to be entirely consistent. The obvious way to test and use this correspondence is to predict curvature effects by using one of the well-known empirical formulae which predict buoyancy effects as a function of Richardson number, and this is done in § 3. The predicted effects of curvature are surprisingly large (10 % change in mixing length in a boundary layer with $\delta/R = 1/300$) and the agreement between an existing calculation method and measurements on curved surfaces is greatly improved. Measurements in a rotating duct by Halleen & Johnston (1967) also support the analogy (§ 4), which should be of general use for rough estimates of the effects of streamline curvature on shear flow development.

2. Curved-flow equivalents of thermal-convection parameters

For ease of reference to the meteorological literature, such as the general introduction of Lumley & Panofsky (1964), the z-axis is taken normal to the surface and the curved-flow parameters in equations ($n\,b$) are given, in quotation marks, the names of their meteorological equivalents in equations ($n\,a$). In the case of small buoyancy or curvature effects the parameters in (i) to (iii) and (v) below all become the same (see (8)) but a discussion of their different physical bases is the simplest way of demonstrating the self-consistency of the analogy.

(i) Elementary analysis shows that the circular frequency of small, vertical, adiabatic oscillations of an element of a perfect gas displaced from its initial height in a temperature-stratified environment is the *Brunt–Väisälä frequency*

$$\omega_{BV} = \left[\frac{g}{T}\left(\frac{\partial T}{\partial z} + \Gamma\right)\right]^{\frac{1}{2}}, \tag{1a}$$

where Γ is the adiabatic lapse rate g/c_p. Neutral static stability corresponds to zero frequency and instability to imaginary frequency.

Similarly, by assuming that an element of fluid displaced radially in a flow with radius of curvature R retains its angular momentum or total pressure it can be shown that the frequency of small oscillations in this case is

$$``\omega_{BV}" = \left[2\frac{U}{R^2}\frac{\partial}{\partial z}(UR)\right]^{\frac{1}{2}} = \left(\frac{2}{\rho R}\frac{\partial P}{\partial z}\right)^{\frac{1}{2}} = \left(2\frac{U}{R}\,(\text{mean vorticity})\right)^{\frac{1}{2}} \tag{1b}$$

for an incompressible flow, where P is the total pressure and R is taken positive if the streamlines are convex upwards, or

$$``\omega_{BV}" = \left[2\frac{U}{R^2}\left(1 + \frac{\gamma - 1}{2}M^2\right)\frac{\partial}{\partial z}\,(UR)\right]^{\frac{1}{2}} = \left(\frac{2}{\rho R}\frac{\partial P}{\partial z}\right)^{\frac{1}{2}} \quad \text{again} \tag{1c}$$

for a compressible flow at constant total temperature, in which the displaced element is supposed to retain its total temperature (the 'strong Reynolds analogy' of Morkovin (1964)) as well as its angular momentum. Rotta (1967) shows that stability in compressible flow depends on the sign of $\partial/\partial z\,(UR)$ by considering the difference between the actual radial density gradient and the 'isentropic' density gradient

$$\frac{\partial \rho}{\partial z} = \frac{\rho}{\gamma p}\frac{\partial p}{\partial z}.$$

If the total temperature of the mean flow is not uniform the frequency depends on the total temperature gradient as well as the angular-momentum gradient. We postpone further discussion of compressible flow to § 5. Equation (1 *b*) was derived for the special case of solid-body rotation by Yeh (see Traugott 1958); here "ω_{BV}" is twice the angular velocity of rotation.

(ii) The *gradient Richardson number* is the ratio of buoyancy to inertia forces and can equally be regarded as the square of the ratio of ω_{BV} to a typical frequency scale of the shear flow, taken as the mean vorticity $\partial U/\partial z$.

Then

$$Ri = \frac{g}{T}\left(\frac{\partial T}{\partial z} + \Gamma\right)\Big/\left(\frac{\partial U}{\partial z}\right)^2, \tag{2a}$$

being *positive* for *stable* flow, and analogously for an incompressible curved flow we can define a 'Richardson number'

$$\text{“}Ri\text{”} = 2\frac{U}{R^2}\frac{\partial}{\partial z}(UR)\Big/\left(\frac{\partial U}{\partial z}\right)^2 = 2S(1+S), \tag{2b}$$

where $S = (U/R)/(\partial U/\partial z)$. For small S, $Ri = 2S$, the dimensionless parameter derived by Prandtl and various later authors (e.g. Stratford, Jawor & Smith 1964, Giles, Hays & Sawyer 1966) by classical mixing-length arguments. The assumption of small S is justifiable in most flows where curvature effects do not dominate but in the unstable rotating-cylinder flow and in the curved-channel flow investigated by Eskinazi & Yeh (1956) the analogy of free convection occurs and $UR \simeq$ constant in the central part of the flow, just as $T \simeq$ constant in the central part of the free convection flow between horizontal parallel planes.

(iii) The *flux Richardson number*, R_f, is minus the ratio of turbulent energy production by buoyancy forces to production by shearing forces,

$$R_f = g\frac{\overline{\theta w}}{T}\Big/\overline{uw}\frac{\partial U}{\partial z}, \tag{3a}$$

where $\overline{\theta w}$ is the thermometric heat flux and $\overline{uw} = -\tau/\rho$. It can be seen from the equations for the mean-square intensity or kinetic energy of each velocity component that the buoyant production goes into the w-component in the first instance while the shear production goes into the u-component: of course pressure fluctuations redistribute the energy so that R_f is not a direct measure of changes in partition of energy. However, R_f is a more meaningful parameter than Ri, because the latter is related to the static stability of an isolated element of fluid in a non-turbulent environment; R_f is at least the ratio of two terms in a universally valid, if intractable, equation for the turbulent kinetic energy $\frac{1}{2}\rho(\overline{u^2}+\overline{v^2}+\overline{w^2}) \equiv \frac{1}{2}\rho\overline{q^2}$. The turbulent Prandtl number, K_M/K_H, is Ri/R_f. In the absence of appreciable transport of turbulent energy by horizontal advection or vertical diffusion, the turbulent energy equation becomes

(shear production) + (buoyant production) = (dissipation)

or
$$\tau(\partial U/\partial z)\,(1-R_f) = \text{dissipation} \tag{4}$$

and most of the meteorological analysis quoted below applies to this special case, which is a good approximation in the inner layer of the atmospheric boundary layer (Busch & Panofsky 1968).

In curved flows (or, more exactly, flows analyzed by a system of cylindrical co-ordinates) production terms occur in the mean-square intensity equations for both the streamwise (u) and radial (w) components. The ratio of (minus) the w-component 'buoyant' production to the u-component 'shear' production is

$$\text{``}R_f\text{''} = \frac{2\overline{uw}\,\dfrac{U}{R}}{\dfrac{\overline{uw}}{R}\dfrac{\partial}{\partial z}(UR)} = \frac{2S}{1+S}, \tag{3b}$$

which, curiously, has the angular momentum gradient in the denominator but reduces to Ri if S is small, the case of most practical interest. In curved flow, we note, there is always a simple relation between Ri and R_f. The 'turbulent Prandtl number' has no useful meaning; it is formally $(1+S)^2$. This expression for R_f has been derived independently by Wyngaard (1967).

(iv) The '*stress Richardson number*' is probably a more meaningful quantity than the flux Richardson number. It is (minus) the ratio of the two production terms in the Reynolds stress equation for $D\overline{uw}/Dt$, rather than the energy equation for $D\overline{q^2}/Dt$. In buoyant flows, this 'stress Richardson number' is

$$R_s = \frac{g\,\dfrac{\overline{\theta u}}{T}}{\overline{w^2}\,\dfrac{\partial U}{\partial z}}, \tag{5a}$$

i.e.
$$R_s = \frac{\overline{\theta u}\,.\,\overline{uw}}{\overline{\theta w}\,.\,\overline{w^2}}\,R_f.$$

In curved flows

$$\text{``}R_s\text{''} = \frac{2\overline{u^2}\,\dfrac{U}{R}}{\dfrac{\overline{w^2}}{R}\dfrac{\partial}{\partial z}(UR)}, \tag{5b}$$

i.e.
$$\text{``}R_s\text{''} = \frac{\overline{u^2}}{\overline{w^2}}\,R_f.$$

In both cases, R_s/R_f is about 4 in near-neutral conditions (using laboratory turbulence data); one would expect it to be *greater* in stable conditions and *less* in unstable conditions. An interesting point is that in the inner layer of a boundary layer $\overline{u^2}$ contains an 'inactive' contribution originating in the outer layer, principally because the inner layer sees the outer layer as an unsteady free stream.

This 'inactive' motion is so called because on a flat surface it does not contribute to the shear stress; but on a curved surface (5*b*) shows that it does. The 'stress Richardson number' concept is useful in other cases where extra 'production' terms occur, such as laterally diverging flows.

(v) *The Monin–Oboukhov length*

$$L \equiv -\frac{(\tau/\rho)^{\frac{3}{2}}}{K}\Big/\left(g\frac{\overline{\theta w}}{T}\right) \tag{6a}$$

is written thus so as to appear as

$$L = \frac{\epsilon L_\epsilon/K}{\text{buoyant production}},$$

where ϵ is the local rate of dissipation of turbulent energy and L_ϵ is the dissipation length parameter, $(|\tau/\rho|)^{\frac{3}{2}}/(\text{dissipation})$, similar to the parameter

$$(\overline{w^2})^{\frac{3}{2}}/(\text{dissipation})$$

used by Townsend (1958). L is constant in the inner layer of the Earth's boundary layer, where the shear stress and heat flux are constant.

Equation (4), the turbulent energy equation with negligible transport terms, can be rewritten

$$-\frac{1}{R_f}+1 = -\frac{KL}{L_\epsilon} \tag{7}$$

so that if buoyancy effects are small

$$KL/L_\epsilon \simeq R_f \simeq 2S.$$

In the inner layer of a boundary layer $L_\epsilon = Kz$ in near-neutral conditions so that

$$z/L \simeq R_f.$$

The analogous curved-flow parameter "L" can be defined similarly as

$$\frac{\text{dissipation}\,.\,L_\epsilon/K}{\text{buoyant production}} = \frac{R}{2K}\frac{(\tau/\rho)^{\frac{1}{2}}}{U}\operatorname{sgn}\tau \tag{6b}$$

($\operatorname{sgn}\tau$ appearing because $\epsilon = (|\tau/\rho|)^{\frac{3}{2}}/L_\epsilon$) and is similarly related to R_f. Taking $U = 20(\tau/\rho)^{\frac{1}{2}}$ as a representative value in the inner layer, "L" $\simeq 0{\cdot}06\,R$.

(vi) Parameters for *small* curvature effects. If transport terms and curvature effects are small it can be seen from the preceding definitions that

$$Ri \simeq R_f \simeq 2S = 2\frac{U}{R}\Big/\frac{\partial U}{\partial z} \simeq \frac{L_\epsilon}{KL} = 2\frac{U}{R}\Big/\frac{(\tau/\rho)^{\frac{1}{2}}}{L_\epsilon} \tag{8}$$

and any of these parameters could be used in empirical formulae for curvature effects. The last parameter has the computational advantage of containing no gradients; physically it is simply twice the ratio of the mean-flow angular velocity to a typical r.m.s. angular velocity of the turbulence (for which $(\tau/\rho)^{\frac{1}{2}}/L_\epsilon$ is a more realistic expression than $\partial U/\partial z$ if the two are different). In the absence of large curvature effects L_ϵ/δ is nearly a universal function of z/δ in attached boundary layers (Bradshaw, Ferriss & Atwell 1967); $L_\epsilon = Kz$ for $z/\delta < 0{\cdot}2$ and $L_\epsilon \simeq 0{\cdot}1\,\delta$ for $z/\delta > 0{\cdot}25$; $L_\epsilon \sim (\text{intermittency})^{\frac{1}{2}}$ in the outermost part of the flow.

3. Application of the analogy

None of the parameters in (8) is a *quantitative* measure of buoyancy effects on turbulent intensity or shear stress because transport terms, though usually small compared with production or dissipation, are not usually negligible; therefore there is no unique critical Richardson number for, say, the suppression of turbulence. If transport *is* negligible, however, the turbulent energy equation reduces to the form (4) or (7) and we find that

$$(\tau/\rho)^{\frac{1}{2}}/(\partial U/\partial z) = L_\epsilon \Big/ \left(1+\frac{L_\epsilon}{KL}\right) = L_\epsilon(1-R_f). \tag{9}$$

In the following discussion we shall give the quantity on the left its common name, 'mixing length', and common symbol, l; like the dissipation length parameter, it is a defined quantity having the dimensions of length but not directly related to any length scale of the turbulent eddies. In the above equation R_f or L_ϵ/L *are* quantitative measures of buoyancy effects but we cannot conclude, for instance, that the ratio of l at given z to its value at the same z with $R_f = 0$, l/l_0 say, is simply $1-R_f$ (so that the turbulence vanishes at $R_f = 1$), because L_ϵ/z may itself depend on R_f.

Townsend (1958) showed that $R_f \ngtr \frac{1}{2}$ in an atmosphere with negligible transport, by considering the conservation equation for $\overline{\theta^2}$. The curved-flow analogue of this equation is the energy equation for $\overline{u^2}$, which contains unknown pressure transfer terms, so there is no analogue of Townsend's result.

The ratio l/l_0 is a quantitative measure of buoyancy effects on turbulent shear stress and many workers have examined its behaviour in the inner layer of the atmospheric boundary layer. No data are available for the outer layer, which is not usually clearly distinguished, and the best that can be done at this stage is to apply the inner-layer formulae to the whole of the boundary layer on a curved surface. Since the correction should be fairly small, the imprecision of the mixing-length concept is not too serious a drawback; Busch & Panofsky (1968) give a good idea of the imprecision of the meteorological data. As Prandtl noted, the mixing length in a curved flow should strictly be defined as

$$\text{“}l\text{''} = (\tau/\rho)^{\frac{1}{2}} R \Big/ \frac{\partial}{\partial z}(UR) = \frac{(\tau/\rho)^{\frac{1}{2}}/(\partial U/\partial z)}{1+S}$$

so that "l" is again equal to $L_\epsilon(1-\text{“}R_f\text{''})$ in the simple case mentioned above (see the discussion of (3*b*)), but any formula relating to "l" can easily be put in terms of the mixing length as it is usually defined.

The Monin–Oboukhov formula for the modification of the apparent mixing length by small buoyancy effects (see Lumley & Panofsky 1964) is one of the earliest of such formulae. Although largely superseded it will suffice for the present simple discussion for small Ri. It is

$$l_0/l \equiv \phi = 1+(\beta z/L) \simeq 1+\beta Ri \simeq 1/(1-\beta Ri). \tag{10a}$$

In the case of curved flow this formula would apply to "l" and the corresponding formula for l would be

$$l_0/l \simeq 1+(\beta-\tfrac{1}{2})\,Ri. \tag{10b}$$

In unstable conditions ($Ri < 0$), β is about 4·5 and in stable conditions it seems to be between 7 and 10; clearly we may neglect the difference between β and $\beta - \frac{1}{2}$.

Note that the simple-minded conclusion that $l/l_0 = (1 - (\text{Richardson number}))$ is much nearer the facts if one inserts the stress Richardson number rather than the flux Richardson number. Since the Richardson number in this formula is supposed to represent the effect of buoyancy on shear stress rather than on turbulent intensity, this is probably more than a coincidence but we do not know enough about the Reynolds stress equation to be sure.

The only direct evidence in favour of a Monin–Oboukhov type of formula in curved flow is the work of Giles *et al.* (1966), who found that the apparent mixing length in a wall jet on a curved surface varied as $l/l_0 = 1 - 6S \simeq 1 - 3Ri$ in the range $-0{\cdot}5 < Ri < 0{\cdot}05$. The Monin–Oboukhov formula is not valid for such large negative Richardson numbers and the formula of 'Keyps' (an acronym of authors' names) is commonly used instead; this is

$$l/l_0 = (1 - 18Ri)^{\frac{1}{4}} \quad \text{for} \quad -0{\cdot}5 < Ri < 0.$$

Near $Ri = 0$ this coincides with the Monin–Oboukhov formula

$$l/l_0 = 1 - 4{\cdot}5Ri$$

but a best straight line fit over the range $-0{\cdot}5 < Ri < 0$ is

$$l/l_0 = 1 - 2Ri$$

so that the 'Coanda effect' result of Giles *et al.* is at least compatible with meteorological data.

There is no direct evidence about the behaviour of β in laboratory boundary layers (we note that the commoner case of a convex wall ($R > 0$) implies *stable* conditions) although several recent authors (e.g. Thompson 1965) have suggested that curvature effects may be appreciable, even in typical aerofoil cases. Apparently, nearly all the authors who have investigated curvature effects chose such highly curved surfaces that the flow was grossly altered; for instance, in the concave wall experiments of Wilcken (1930), Patel (1965) and Mackrodt (1967), and in the curved-channel experiment of Eskinazi & Yeh (1956), the curved-flow analogue of free convection appeared. We cannot expect the present analogy to apply quantitatively to such flows. Even if $|\delta/R|$ is as small as 1/300, a value which may easily be reached on thick aerofoils or turbomachine blades, the Monin–Oboukhov formula with $\beta = 7$ indicates a change of about 10 % in apparent mixing length in the outer part of a boundary layer (taking $U = 0{\cdot}8U_1$ and $\partial U/\partial z = 0{\cdot}4U_1/\delta$). It is noteworthy that if $\delta/R = 1/300$ the static pressure change across a typical boundary layer is less than one-half per cent of the free-stream dynamic pressure; the effect of curvature on the turbulence greatly exceeds the effect on the mean-motion equations.

As a first attempt at using the ideas of this paper, we have applied the Monin–Oboukhov formula for the mixing length to the *dissipation length parameter* L_ϵ used in the calculation method of Bradshaw *et al.* (1967), neglecting the buoyant production. This is roughly the same as applying the formula to the mixing length throughout the boundary layer; the effects of curvature on the other empirical functions used in the calculation method have been ignored. We have replaced L_ϵ by

$$L_\epsilon \Big/ \left(1 + 7\left(2\frac{L_\epsilon}{R}\frac{U}{(\tau/\rho)^{\frac{1}{2}}}\right)\right)$$

using the form of 'Richardson number' recommended in §2(vi) and taking $\beta = 7$. The effect of the curvature correction on L_ϵ in a boundary layer in zero pressure gradient with $\delta/R = 1/80$ is shown in figure 1. Since the velocity at

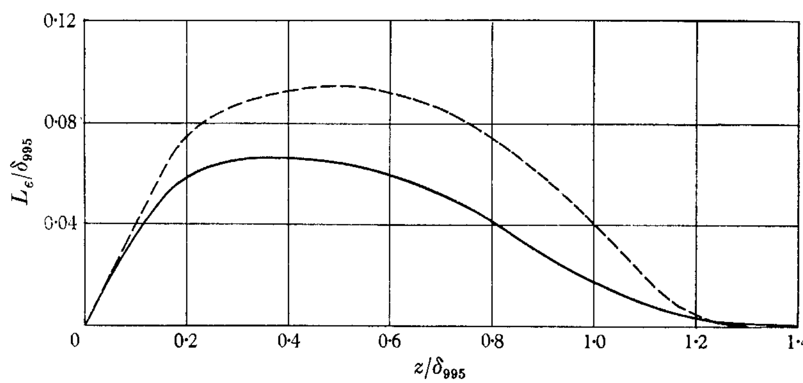

FIGURE 1. Effect of curvature on dissipation length parameter. — — —, flat surface; ———, $\delta/R = 1/80$, zero pressure gradient, $U_1\delta_2/\nu \simeq 15{,}000$, $\beta = 7$. δ_{995} is the distance from the surface at which $U/U_1 = 0{\cdot}995$.

$y/\delta = 0{\cdot}2$ is as high as 0·8 of the free-stream value the large changes in L_ϵ in the outer layer do not produce proportionate changes in, say, the skin friction coefficient c_f; in the case shown, the calculated c_f was 0·89 times the value for a flat surface at the same momentum-thickness Reynolds number $U_1\delta_2/\nu$, about 15,000. In strongly retarded boundary layers the outer layer exerts a greater influence but the effect of curvature on L_ϵ is less, according to the simple formula, because $U/(\tau/\rho)^{\frac{1}{2}}$ is less. However the use of the Monin–Oboukhov formula in the *outer* layer is justifiable only by the most naive mixing-length arguments, even if one accepts the plausibility of the basic analogy between curvature and buoyancy effects; a more refined approach would have to include the effect of curvature on the large eddies, as expressed by some outer-layer average Richardson number like $2\delta U_1/(R(\tau_{\max}/\rho)^{\frac{1}{2}})$, which would change the energy diffusion as well as L_ϵ.

The only well-authenticated test cases with prolonged regions of significant curvature are the aerofoil of Schubauer & Klebanoff (1951), with $R = 9{\cdot}4$ m downstream of the pressure minimum, where $\delta = 6{\cdot}4$ cm, and the experiment of Schmidbauer (1936) on a convex surface of radius 150 cm (the boundary-layer thickness at $x = 46$ cm being 1·1 cm) with a pressure gradient changing from mildly adverse to strongly favourable. The results of calculations by the method of Bradshaw *et al.* with and without the simple curvature correction are shown in figures 2 and 3. In both cases the 'experimental' surface shear stress has been

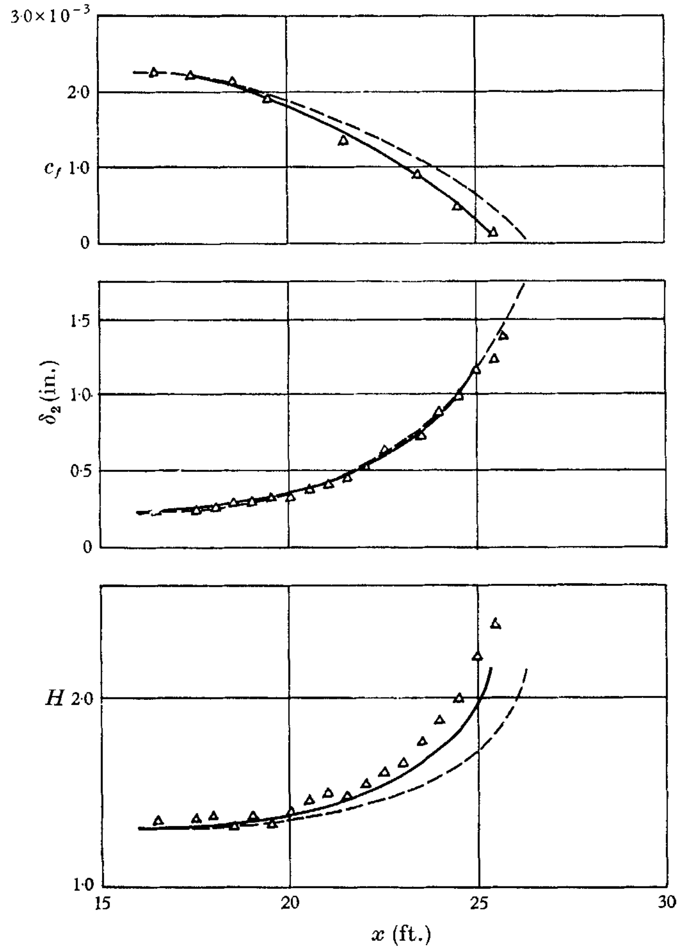

FIGURE 2. The boundary layer of Schubauer & Klebanoff (1951). △, experiment; — — —, calculation without curvature correction; ———, calculation with curvature correction. H is the ratio of displacement thickness δ_1 to momentum thickness δ_2.

inferred from the logarithmic law close to the surface and the imbalance in the momentum integral equation has been used to deduce the apparent spanwise convergence of the experimental flow; the calculations were then carried out with the same distribution of convergence. The convergence values, as used by Thompson (1964), were kindly supplied by the author; those used for the Schubauer & Klebanoff runs differ slightly from the values used by Bradshaw *et al.* The assumed initial shear stress profiles are obviously slightly wrong in both cases, particularly that of Schmidbauer, since the initial trend of H is wrong; this, of course, has nothing to do with the reliability of the calculation method as such. The improvement in the calculated results obtained by using the simple curvature correction is very encouraging and suggests that the correction should be adequate for most aerofoil calculations, pending a better understanding of the effect of buoyancy or curvature on the turbulence structure.

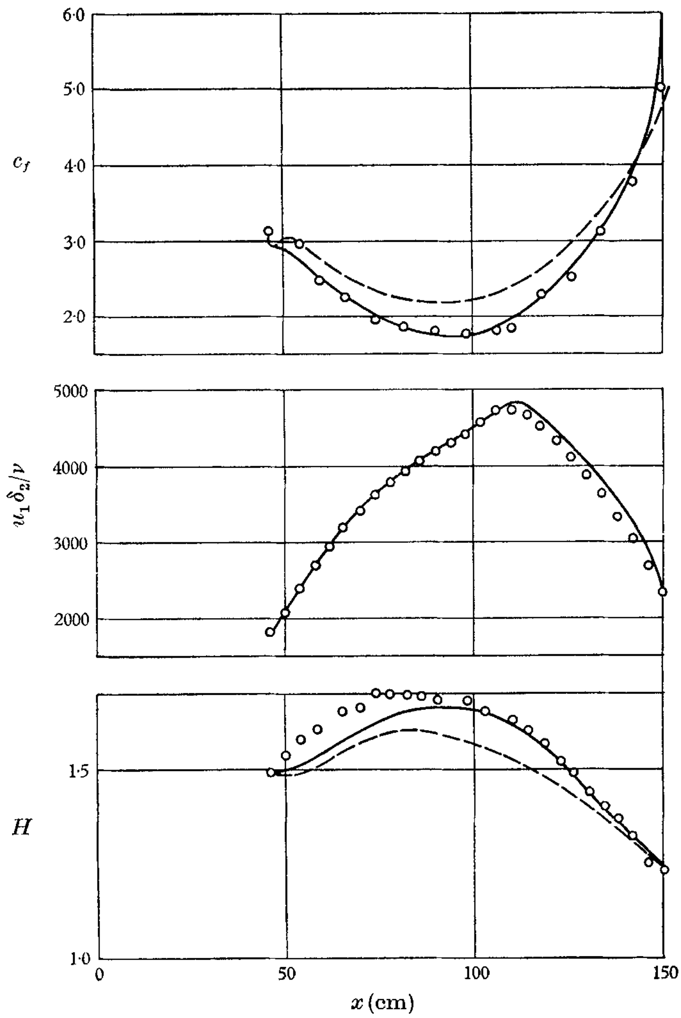

FIGURE 3. The boundary layer of Schmidbauer (1936). ○, experiment; — — —, calculation without curvature correction; ———, calculation with curvature correction.

4. Rotating flows

The analysis and discussion of §§2 and 3 were carried through for flow over fixed curved surfaces, for simplicity, but exactly the same arguments can be applied to shear layers in rotating fluids, especially rotation about a spanwise axis (i.e. an axis perpendicular to the plane of the mean rate of strain). In this case the 'centrifugal force' can be absorbed in the pressure gradient term if the flow is analyzed with respect to axes fixed in the body and the only extra term in the equations of motion is the apparent Coriolis force $\mathbf{V} \times \mathbf{\Omega}$, which, to the boundary-layer approximation, reduces to a force $2\Omega U$ in the z direction, corresponding to the force U^2/R in the z direction in flow over a curved surface. To emphasize this correspondence we take Ω to be positive clockwise in the (x, z)-plane.

The 'Brunt–Väisälä frequency' is given by

$$\omega_{BV}^2 = 2\Omega\left(\frac{\partial U}{\partial z}+2\Omega\right) \tag{1c}$$

so that the gradient 'Richardson number' is

$$Ri = 2\Omega\left(\frac{\partial U}{\partial z}+2\Omega\right)\bigg/\left(\frac{\partial U}{\partial z}\right)^2 = S(1+S) \tag{2c}$$

where $S = 2\Omega/(\partial U/\partial z)$.

The turbulent energy equation given by Halleen & Johnston (1967) has a v-component production term $2\Omega\tau$ and a u-component production term

$$\tau\,\partial U/\partial z - 2\Omega\tau.$$

Therefore the flux Richardson number is

$$R_f = \frac{2\Omega}{\partial U/\partial z+2\Omega} = S/(1+S) = \frac{\pm\sqrt{(1+4Ri)}-1}{\pm\sqrt{(1+4Ri)}+1}. \tag{3c}$$

The positive square root must be taken for $S > -\frac{1}{2}$, and conversely. As in the case of curved flow, R_f reduces to Ri for small 'buoyancy'. The stress Richardson number is

$$R_s = (\overline{u^2}/\overline{w^2})\,R_f \tag{5c}$$

as in curved flow. The Monin–Oboukhov length is

$$L = \{(\tau/\rho)^{\frac{1}{2}}/2\Omega K\}\,\mathrm{sgn}\,\tau \tag{6c}$$

so that

$$\frac{L_\epsilon}{L} = \frac{2K\Omega L_\epsilon}{(\tau/\rho)^{\frac{1}{2}}}\,\mathrm{sgn}\,\tau.$$

The Monin–Oboukhov formula in this case can be integrated exactly, as in a stable atmosphere. If we put $Ri = 2\Omega/(\partial U/\partial z)$ and use $l/l_0 = 1-\beta Ri$ the mixing-length formula gives

$$U = \int\frac{(\tau/\rho)^{\frac{1}{2}}}{l_0}\,dz + 2\beta\Omega z$$

or

$$U = \{(\tau_w/\rho)^{\frac{1}{2}}/K\}\,(\log z + \text{constant}) + 2\beta\Omega z$$

in the inner layer where $\tau = \tau_w$ and $l_0 = Kz$.

A rough estimate of the value of β can be obtained from Halleen & Johnston's measurements in fully developed flow in a rectangular duct rotating about an axis parallel to the longer side of the rectangle, by plotting the deviations from the logarithmic profile (in a stationary duct the logarithmic profile extends nearly to the centre although the conditions $\tau = \tau_w$ and $L_\epsilon = Kz$ are certainly not satisfied at more than about 0·1 duct heights from each wall). The results are shown in figure 4; β is about 4 on the stable side and 2 on the unstable side. Although these values are gratifyingly *consistent*, they are not quantitatively useful because the straight lines in figure 4 are based on the deviations in the range $200 < u_\tau z/\nu < 400$ whereas the analysis applies only to the inner layer

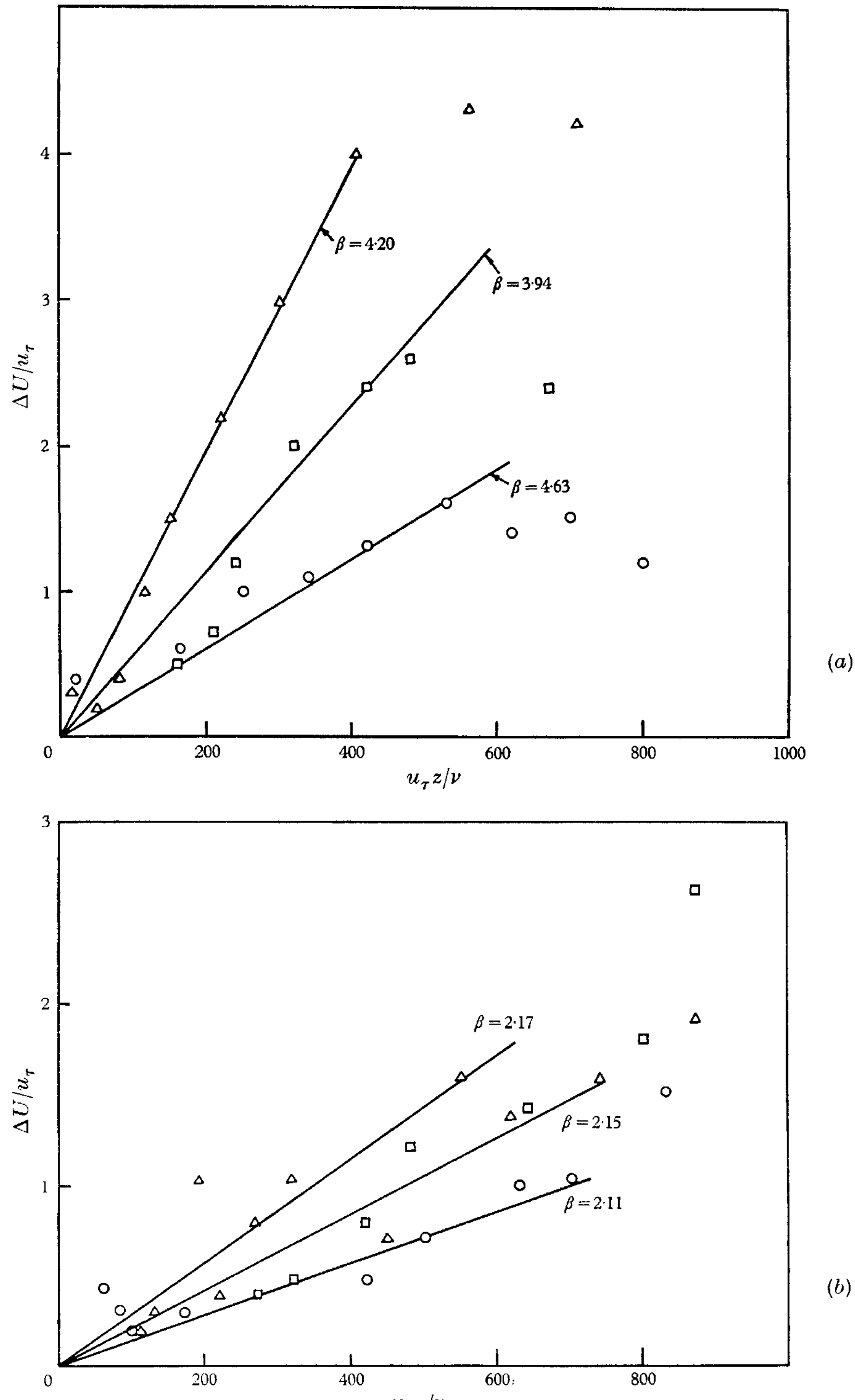

FIGURE 4. Departure from logarithmic profile in rotating duct flow (Halleen & Johnston 1967). (*a*) Stable: ○, $\Omega\nu/u_\tau^2 = 0{\cdot}00328$; □, $0{\cdot}00379$; △, $0{\cdot}0120$. (*b*) Unstable: ○, $\Omega\nu/u_\tau^2 = 0{\cdot}00269$; □, $0{\cdot}00510$; △, $0{\cdot}00738$.

which ends at $u_\tau z/\nu$ less than 200. More experiments would be needed before one could say definitely that β was smaller in rotating flow than in curved flow; one would expect less difference between rotation and curvature than between curvature and buoyancy, because both Coriolis and 'centrifugal' forces depend on U-component velocity whereas buoyancy forces depend on temperature.

Combined curvature and rotation effects are found in axisymmetric rotating flows; an example is the boundary layer on a rotating cylinder in an axial stream investigated by Parr (1963). The combined Richardson number is found to be

$$2\frac{U_m}{R}\frac{\lambda^2(1-U/U_m)}{(1+\lambda^2)(\partial U/\partial z)} \sim 2\frac{U_m}{R}\frac{\{1-U/U_m\}}{\partial U/\partial z} \quad \text{for large } \lambda,$$

where U_m is the free-stream speed, U the axial component of velocity, z the radial co-ordinate and $\lambda = \Omega R/U_m$. The Richardson number is zero in the axial free stream, as we should expect, and in general the effects of curvature and rotation oppose each other. It should be noted that this is not a truly three-dimensional flow; the velocity profiles measured with respect to the cylinder are collinear and the chief effect of rotation is simply to increase the length of the (helical) path travelled by the fluid from one end of the cylinder to the other; the boundary-layer thickness at the downstream end of Parr's rotating forebody is indeed very nearly proportional to this path length, $x\sqrt{(1+\lambda^2)}$, so that the net dynamical effects of rotation and curvature were small in this experiment. According to the present analysis, the velocity gradient that determines the Brunt–Väisälä frequency is $\partial V/\partial z$, where V is the circumferential component, whereas the velocity gradient that represents a typical turbulence frequency is the resultant $\partial(U^2+V^2)^{\frac{1}{2}}/\partial z$; this accounts for the odd factors in λ ($\equiv \tan^{-1} V/U$) in the Richardson number. I am grateful to Dr T. S. Cham of Cambridge University Engineering Department for a discussion of Parr's flow.

5. Compressible flow

Rotta (1967) used the turbulent energy equation to derive a formula for curvature effects in a compressible flow. Rotta neglects the difference between l and "l" and assumes that L_ϵ is unaltered, so that his final result is in error, but he makes the important point that turbulent energy is produced by the product of turbulent mass flux $\overline{\rho' w}$ and mean acceleration in the z direction, U^2/R. The effect is to multiply R_f by a factor $1+\frac{1}{2}(\gamma-1)M^2$; it can be seen from (1$c$) that Ri is similarly altered (in a first draft of this paper I wrongly said that Rotta's factor $1+(\gamma-1)M^2$ was not directly related to the factor in (1c); it is, because Rotta evaluated $\overline{\rho' w}$ by using the 'strong Reynolds analogy', but Rotta's factor is wrong, for the reasons mentioned above). The implication is that curvature effects can be much larger at high supersonic speeds, and Thomann (1968) has found large changes in heat transfer at $M = 2{\cdot}5$ on curved surfaces. In high-temperature low-speed flows (Barrow 1968) the factor $1+\frac{1}{2}(\gamma-1)M^2$ is replaced by $T/T_{\text{free stream}}$ approximately.

6. Conclusions

A formally exact analogy can be drawn between meteorological parameters and the parameters describing the effect on a turbulent flow of streamline curvature (in the plane of the principal rate of strain) or rotation (about an axis normal to that plane). A fairly close analogy between the *phenomena* is implied by the close correlation between temperature fluctuations (which produce buoyancy fluctuations) and u-component velocity fluctuations (which produce centrifugal or Coriolis 'force' fluctuations).

A curved-flow form of the Monin–Oboukhov formula for the change of apparent mixing length with Richardson number is found to be reasonably consistent with the limited data and its use to calculate the change of dissipation length parameter (a generalized mixing length) produces a great improvement in the agreement between prediction and experiment in boundary layers on curved walls.

However, the meteorological formulae relate only to the inner layer of the Earth's boundary layer and it is likely that the effects of curvature or buoyancy on the outer layer of a boundary layer will be rather different because of the effects of body forces on the processes of turbulent energy diffusion and on the large eddy structure in general. More experimental work is needed to investigate this point and to establish whether the extra energy production terms that occur in curved compressible flow (Rotta 1967) cause curvature effects to increase rapidly with Mach number. This last point is particularly important in view of the frequent occurrence of highly curved surfaces in engine intakes and other supersonic flows.

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